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Low-energy gluon contributions to the vacuum polarization of heavy quarks

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Abstract

We calculate a correction to the electromagnetic current induced by a heavy quark loop. The contribution of this correction to the vacuum polarization function appears at the $O(\alpha_s^3)$ order of perturbation theory and has a qualitatively new feature – its absorptive part starts at zero energy in contrast to other contributions where the absorptive parts start at the two-particle threshold. Our result imposes a constraint on the order n of the moments used in the heavy-quark sum rules, $n < 4$.

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High precision tests of the standard model remain one of the main topics of particle phenomenology. The recent observation of a possible signal from the Higgs may complete the experimentally confirmed list of the standard model particles [1]. Because experimental data are becoming more and more accurate, the determination of numerical values of the parameters of the standard model lagrangian will require more accurate theoretical formulae. Recently an essential development in high-order perturbation theory calculations has been observed. A remarkable progress has been made in the heavy quark physics where a number of new physical effects have been described theoretically with high precision. The cross section of top–antitop production near the threshold has been calculated at the next-to-next-to-leading order of an expansion in the strong coupling constant and velocity of a heavy quark with an exact account for Coulomb interaction (as a review, see Ref. [2]). This allows for the best determination of a numerical value of the top quark mass. The method of Coulomb resummation resides on a nonrelativistic approximation for the Green function of the quark-antiquark system near the threshold and has been successfully used for the heavy quark mass determination within the sum rules [3, 4, 5]. Being applied to quarkonium systems this method is considered to give the best estimates of heavy quark mass parameters [6, 7, 8, 9]. Technically an enhancement of near-threshold contributions to sum rules is achieved by considering integrals of the spectral density of the heavy quark production with weight functions which suppress the high-energy tail of the spectrum. The integrals with weight functions $1/s^n$ for different positive integer n, $s = E^2$, where E is the total energy of the quark-antiquark system, are called moments of the spectral density and most often used in the sum rules analysis [10].

In the present note we show that there is a strong constraint on the order n of the moment that can be used in heavy quark sum rules. Because of the contribution of low-energy gluons, only moments with $n < 4$ exist if theoretical expressions for the corrrelators include the $O(\alpha_s^3)$ order of perturbation theory.

The basic quantity for the analysis within sum rules is a vacuum polarization function $\Pi(q^2)$

$$12\pi^2 i \int \langle T j_\mu(x) j_\nu(0)\rangle e^{iqx} d^4x = (q_\mu q_\nu - g_{\mu\nu} q^2)\Pi(q^2) \tag{1}$$

of the vector current $j^\mu = \bar{q}\gamma^\mu q$ of a heavy fermion q. With the spectral density $\rho(s)$ defined by the relation

$$\rho(s) = \frac{1}{2\pi i}(\Pi(s + i0) - \Pi(s - i0)), \quad s > 0 \tag{2}$$

the dispersion representation

$$\Pi(q^2) = \int \frac{\rho(s)ds}{s - q^2} \tag{3}$$

holds. A necessary regularization (subtractions, for instance) is assumed in Eq. (3). The normalization of the vacuum polarization function $\Pi(q^2)$ in Eq. (1) is chosen such that one

obtains a high-energy limit $\lim_{s\to\infty}\rho(s) = 1$ for a lepton. For the quark in the fundamental representation of the $SU(N_c)$ gauge group a high-energy limit of the spectral density reads $\rho(\infty) = N_c$. The integral in Eq. (3) runs over the whole spectrum of the correlator in Eq. (1) or over the whole support of the spectral density $\rho(s)$ in Eq. (2). The moments of the spectral density $\rho(s)$ of the form

$$\mathcal{M}_n = \int \frac{\rho(s)ds}{s^{n+1}} \tag{4}$$

are usually studied within the sum rules method for heavy quarks [10]. These moments are related to the derivatives of the vacuum polarization function $\Pi(q^2)$ at the origin,

$$\mathcal{M}_n = \frac{1}{n!}\left(\frac{d}{dq^2}\right)^n \Pi(q^2)\Big|_{q^2=0}. \tag{5}$$

Such moments are chosen in order to suppress a high energy part of the spectral density $\rho(s)$ which is not measured accurately in the experiment. Within the sum rule method one assumes that the moments in Eq. (4) can be calculated for any n or, equivalently, that the derivatives in Eq. (5) exist for any n. The existence of moments seems to be obvious because one implicitly assumes that the spectral density of the heavy quark electromagnetic currents $\rho(s)$ vanishes below the two-particle threshold $s = 4m^2$, which means that the vacuum polarization function of heavy quarks $\Pi(q^2)$ is analytic in the whole complex plane of q^2 except for the cut along the positive real axis starting from $4m^2$. This assumption about the analytic properties of the vacuum polarization function $\Pi(q^2)$ is known to be wrong if a resummation of Coulomb effects to all orders of perturbation theory is performed: as a result of such a resummation the Coulomb bound states appear below the perturbation theory threshold $s = 4m^2$. The assumption that the moments in Eq. (4) exist for any n may also be wrong in high orders of perturbation theory in models with massless particles, for example, in QCD with massless gluons. The validity of this assumption depends on details of the interaction. In QCD, at the $O(\alpha_s^3)$ order of perturbation theory there is a contribution of massless states to the correlator in Eq. (1) that leads to the infrared (small s) divergence of moments for large n because of the branching point (cut) singularity of $\Pi(q^2)$ at the origin. We determine the behaviour of the vacuum polarization function $\Pi(q^2)$ at small q^2 ($q^2 \ll m^2$) as

$$\Pi(q^2)|_{q^2\approx 0} = C_g\left(\frac{q^2}{4m^2}\right)^4 \ln\left(\frac{\mu^2}{-q^2}\right) \tag{6}$$

with

$$C_g = \frac{17}{243000} d_{abc}d_{abc}\left(\frac{\alpha_s}{\pi}\right)^3. \tag{7}$$

Here d_{abc} are the totally symmetric structure constants of the $SU(N_c)$ gauge group defined by the relation $d_{abc} = 2\mathrm{tr}(\{t^a, t^b\}t^c)$, and t^a are generators of the group with normalization

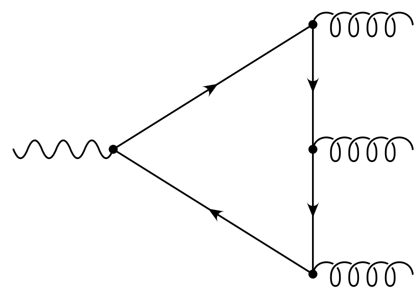

Figure 1: Heavy quark loop correction to the electromagnetic current

$\mathrm{tr}(t^a t^b) = 1/2$. For the $SU(3)$ gauge group of QCD one has $d_{abc}d_{abc} = 40/3$. The parameter μ in Eq. (6) is the renormalization point.

The singularity of the vacuum polarization function given in Eq. (6) (a cut along the positive real axis in a complex q^2-plane) prevents one from calculating moments of the spectral density in Eq. (4) with $n > 4$. Indeed, the high order derivatives of $\Pi(q^2)$ at the origin determining the high order moments according to Eq. (5) do not exist for $n > 4$ because of a branching point singularity as one can see from Eq. (6). In terms of the moments one can see this by calculating the behaviour of the spectral density at small energy s,

$$\rho(s)|_{s\approx 0} = C_g \left(\frac{s}{4m^2}\right)^4 \tag{8}$$

which makes integrals in Eq. (4) divergent at small s for $n > 4$. The formulae for the vacuum polarization function in Eqs. (6) and (7) are given for a heavy quark in the $SU(N_c) \otimes U(1)$ gauge model. The result for QED may be obtained with the obvious substitution $\alpha_s \to \alpha$ for the coupling constant and by setting $d_{abc}d_{abc} = 1$. Contributions of light (massless) quarks appear in the $O(\alpha_s^4)$ order of perturbation theory and are neglected.

We now present a derivation of our result given in Eqs. (6) and (7) and discuss some consequences for the phenomenology of heavy quarks. Note that the induced current is a correction of order $1/m^4$ in the inverse heavy quark mass which vanishes in the limit of an infinitely heavy quark. Corrections in inverse heavy quark masses are important for tests of the standard model at the present level of precision and have been already discussed in various areas of particle phenomenology [11, 12, 13].

A diagram which gives a correction to the electromagnetic current due to a heavy quark loop is given in Fig. 1. Two-gluon transitions are forbidden according to a generalization of Farry's theorem to nonabelian theories [14]. We are interested in a behaviour of the amplitude associated with the diagram Fig. 1 at low energy and therefore take the limit of a very heavy quark. Formally the limit $m \to \infty$ is taken which in physical terms means that m is much larger than all momenta of external legs of the diagram, namely the three gluons and the photon. The

induced current J^μ is written in a covariant form as a derivative of an antisymmetric operator $\mathcal{O}_{\mu\nu}$ built from the gluon fields only,

$$J_\mu = \partial_\nu \mathcal{O}_{\mu\nu}, \qquad \mathcal{O}_{\mu\nu} + \mathcal{O}_{\nu\mu} = 0. \tag{9}$$

This structure of the induced current automatically guarantees the current conservation

$$\partial_\mu J^\mu = 0 \tag{10}$$

as it should be for the electromagnetic current.

A straightforward calculation of the diagram presented in Fig. 1 gives the result for the induced correction to the electromagnetic current

$$J^\mu = \frac{-g_s^3}{1440\pi^2 m^4}\left(5\partial_\nu \mathcal{O}_1^{\mu\nu} + 14\partial_\nu \mathcal{O}_2^{\mu\nu}\right) \tag{11}$$

with

$$\mathcal{O}_1^{\mu\nu} = d_{abc} G^a_{\mu\nu} G^b_{\alpha\beta} G^c_{\alpha\beta}, \quad \mathcal{O}_2^{\mu\nu} = d_{abc} G^a_{\mu\alpha} G^b_{\alpha\beta} G^c_{\beta\nu} \tag{12}$$

where $G^a_{\mu\alpha}$ is a gauge field strength tensor for the gauge group $SU(N_c)$.

A correlator of the induced current J^μ has a general form

$$\langle T J_\mu(x) J_\nu(0)\rangle = -\partial_\alpha \partial_\beta \langle T \mathcal{O}_{\mu\alpha}(x) \mathcal{O}_{\nu\beta}\rangle \tag{13}$$

where an explicit expression of the current as a derivative of the antisymmetric operator $\mathcal{O}_{\mu\nu}$ has been employed. The resulting correlator $\langle T \mathcal{O}_{\mu\alpha}(x) \mathcal{O}_{\nu\beta}\rangle$ in Eq. (13) contains only gluonic operators as is seen from Eqs. (11) and (12). Such correlators were considered previously in the framework of perturbation theory [15, 16]. In the leading order of perturbation theory the correlator in Eq. (13) has a topological structure of a sunset diagram. Technically, a convenient procedure of computing the sunset-type diagrams is to work in the configuration space [17]. We find

$$\langle T J_\mu(x) J_\nu(0)\rangle = -\frac{34}{2025\pi^4 m^8}\left(\frac{\alpha_s}{\pi}\right)^3 d_{abc} d_{abc} \left(\partial_\mu \partial_\nu - g_{\mu\nu}\partial^2\right)\frac{1}{x^{12}} \tag{14}$$

A Fourier transform of the correlator in Eq. (14) gives the vacuum polarization function in momentum space which reads

$$12\pi^2 i \int \langle T J_\mu(x) J_\nu(0)\rangle e^{iqx} d^4x = C_g (q_\mu q_\nu - g_{\mu\nu} q^2)\left(\frac{q^2}{4m^2}\right)^4 \ln\left(\frac{\mu^2}{-q^2}\right) \tag{15}$$

with the constant C_g taken from Eq. (7). The spectral density of the polarization function in Eq. (15) is given in Eq. (8).

Note that the spectral density of the correlator in Eq. (14) can be found without an explicit calculation of its Fourier transform. Instead one can use a spectral decomposition (dispersion representation) in configuration space,

$$\frac{i}{x^{12}} = \frac{\pi^2}{2^8 \Gamma(6)\Gamma(5)} \int_0^\infty s^4 D(x^2, s) ds \tag{16}$$

with $D(x^2, s)$ being the propagator of a scalar particle of mass $\sqrt{s}$,

$$D(x^2, m^2) = \frac{im\sqrt{-x^2}K_1(m\sqrt{-x^2})}{4\pi^2(-x^2)} \tag{17}$$

where $K_1(z)$ is a McDonald function (a modified Bessel function of the third kind, see e.g. Ref. [18]). $\Gamma(z)$ is Euler's gamma function.

An asymptotic behaviour of the spectral density of the corresponding contribution for large energy (when the limit of massless quarks can be used) is well known [19, 20, 21] and reads

$$\left(\frac{\alpha_s}{\pi}\right)^3 \frac{d_{abc}d_{abc}}{1024}\left(\frac{176}{3} - 128\zeta(3)\right). \tag{18}$$

Here $\zeta(z)$ is the Riemann ζ function and $\zeta(3) = 1.20206\ldots$. The contribution to the spectral density given in Eq. (18) is negative while our result given in Eq. (8) is positive as it should be for the spectral density of the electromagnetic current which is an Hermitian operator.

In QCD we find

$$\Pi_{\rm QCD}(q^2)|_{q^2\approx 0} = \frac{17}{18225}\left(\frac{\alpha_s}{\pi}\right)^3\left(\frac{q^2}{4m^2}\right)^4 \ln\left(\frac{\mu^2}{-q^2}\right). \tag{19}$$

Our result has an immediate application to the determination of heavy quark parameters within the method of sum rules. Because of the low-energy gluon contributions, the large n ($n > 4$) moments of the spectral density do not exist and cannot be used for phenomenological analyses. Note that in early considerations of sum rules quite large n were used. For instance, the numerical value of the gluon condensate was extracted from sum rules for the moments with $n \sim 10 \div 20$ [10, 22]. In view of our result one has either to limit the accuracy of theoretical calculations for the moments to the $O(\alpha_s^2)$ order of perturbation theory which seems insufficient for a high precision analysis of quarkonium systems (especially if the Coulomb resummation in all orders is performed) or to use only a few first moments with $n < 4$. For small n, however, the high-energy contribution, which is not known experimentally with a reasonable precision, is not sufficiently suppressed and introduces a large quantitative uncertainty into sum rules for the moments. An alternative analysis based on finite energy sum rules is free from such a problem and can be used in phenomenological applications [23].

Note in passing that there is no low-energy gluon contribution (and low-energy divergence problem) for correlators of the currents containing only one heavy quark with mass m. The spectrum of such correlators starts at m^2 and there are no massless intermediate states contributing to the correlator in perturbation theory. The theoretical expressions for such correlators can be used for high precision tests when the accuracy of experimental data in correspondent channels will improve in the future.

To conclude, we have presented a correction to the electromagnetic current of a heavy quark induced by a virtual heavy quark loop. The spectrum of the correlator of such an induced

current starts at zero energy. This fact makes impossible the standard analysis of the moment sum rules for $n > 4$ at the $O(\alpha_s^3)$ order of perturbation theory.

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